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B.Sc. Part-II, Paper IV

DIFFERENTIAL EQUATION (Sum on Singular Solution)

① Examine the equation $4xp^2 = (3x-a)^2$ for a singular solution.

Soln:- Given equation also written as

$$4x\left(\frac{dy}{dx}\right)^2 = (3x-a)^2 \qquad \left[\because p = \frac{dy}{dx}\right]$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{(3x-a)^2}{4x} \Rightarrow \frac{dy}{dx} = \pm\frac{(3x-a)}{(2\sqrt{x})}$$

$$\Rightarrow \frac{dy}{dx} = \pm\left(\frac{3}{2}\sqrt{x} - \frac{a}{2\sqrt{x}}\right)$$

$$\Rightarrow \int dy = \pm\int\left(\frac{3}{2}\sqrt{x} - \frac{a}{2\sqrt{x}}\right)dx$$

$$\Rightarrow y + c = \pm\left\{\frac{3}{2}\times\frac{2}{3}x^{2/3} - \frac{a}{2}\times 2\sqrt{x}\right\}$$

$$= \pm\left\{x^{2/3} - a\sqrt{x}\right\} = \pm\sqrt{x}(x-a)$$

$$\Rightarrow (y+c)^2 = x(x-a)^2.$$

Clearly c-discriminant $\equiv x(x-a)^2 = 0$

and p-discriminant $\equiv x(3x-a)^2 = 0$

Hence the singular solution is $x = 0$ which is contained in both c and p discriminants.

Remark: (i) $x - a = 0$ which occurs twice in c-discriminant and is not included in p-discriminant is the nodal locus.

(ii) $3x - a = 0$ which occurs twice in p-discriminant and is not included in c-discriminant represents tac locus.

(2)

(2) Obtain the primitive and singular solution of the equation $xp^2 - 2yp + 4x = 0$

Soln- Given $xp^2 - 2yp + 4x = 0 \Rightarrow 2yp = xp^2 + 4x$

$\Rightarrow y = \frac{px}{2} + \frac{2x}{p}$ The equation is solvable for y.

Differentiating w.r.t. x, we get

$$p = \frac{1}{2}\left\{p.1 + x\frac{dp}{dx}\right\} + 2\left\{\frac{1}{p}.1 + x\left(-\frac{1}{p^2}\right)\frac{dp}{dx}\right\} \quad \left[\frac{dy}{dx} = p\right]$$

$$\Rightarrow 2p = p + x\frac{dp}{dx} + 4\left\{\frac{1}{p} - \frac{x}{p^2}\frac{dp}{dx}\right\} = p + x\frac{dp}{dx} + \frac{4}{p} - \frac{4x}{p^2}\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx}\left\{x - \frac{4x}{p^2}\right\} + \left\{\frac{4}{p} - p\right\} = 0$$

$$\Rightarrow \frac{dp}{dx}x\left\{1 - \frac{4}{p^2}\right\} + \left\{\frac{4 - p^2}{p}\right\} = 0$$

$$\Rightarrow x\frac{dp}{dx}\left\{\frac{p^2 - 4}{p^2}\right\} - \left\{\frac{p^2 - 4}{p}\right\} = 0$$

$$\Rightarrow \left(\frac{p^2 - 4}{p}\right)\left\{\frac{x}{p}\frac{dp}{dx} - 1\right\} = 0 \Rightarrow \frac{x}{p}\frac{dp}{dx} - 1 = 0$$

$$\Rightarrow \frac{x}{p}\frac{dp}{dx} = 1 \Rightarrow \frac{dp}{dx} = \frac{p}{x} \Rightarrow \frac{dp}{p} = \frac{dx}{x}$$

$$\Rightarrow \int\frac{dp}{p} = \int\frac{dx}{x} \Rightarrow \log p = \log x + \log c$$

$\Rightarrow \log p = \log xc \quad \therefore p = cx$.

Eliminating p, we get the complete primitive as

$$x.c^2x^2 - 2y.cx + 4x = 0 \Rightarrow c^2x^3 - 2cyx + 4x = 0$$

$$\Rightarrow c^2x^2 - 2cy + 4 = 0$$

$\therefore$ c-discriminant is $4y^2 - 4x^2.4 = 0$ $[EN^2C^3]$

i.e. $y^2 - 4x^2 = 0$.

Also p-discriminant is $y^2 - 4x^2 = 0$ $[ET^2C]$

Clearly $y^2 - 4x^2 = 0 \Rightarrow y^2 = 4x^2 \Rightarrow y = \pm 2x$

which occurs once in both p and c- discriminants is the singular solution.